

Direct Determination of Harmonic and Inter-modulation distortions with an application to Single Mode Laser Diodes

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Abstract

Harmonic and Intermodulation distortions occur when a physical system is excited with a single or several frequencies and when the relationship between the input and output is non-linear. Working with non-linearities in the Frequency domain is not straightforward specially when the relationship between the input and output is not trivial. We outline the complete derivation of the Harmonic and Intermodulation distortions from basic principles to a general physical system. For illustration, the procedure is applied to the Single Mode laser diode where the relationship of input to output is non-trivial. The distortions terms are extracted directly from the Laser Diode rate equations and the method is tested by comparison to many results cited in the literature. This methodology is general enough to be applied to the extraction of distortion terms to any desired order in many physical systems in a general and systematic way.

Keywords: Non-linear distortions. Optoelectronic devices. Solid-state lasers.

I. INTRODUCTION

Harmonic and Intermodulation distortions occur when a signal at a single frequency or a superposition of several signals with different frequencies propagate in a non-linear physical system. Historically it was first highlighted in the Radar and Radio Communications industry because of the interest in understanding interference from other radars, jammers, other transmitters or modulators... Presently, it is pervading High Energy Physics particle detectors, wideband amplifiers, satellite communications [1] and other areas of science and technology. To cite some of the areas of interest, it occurs in many types of devices such as Mechanical [2], Acoustical (Microphones for instance), Electronic [3] and Microelectronic [4], Microwave [5], Optical [6], Magnetic [7] and Superconducting [8].

Parasitic frequency terms appear either at integer multiple of the base frequency (harmonic) or as a mixture of two or several multiples of base frequencies (intermodulation) when several base frequencies are used (as in modulation systems for instance). These terms can be either post-filtered or the signal can be pre-distorted in order to avoid the appearance of these unwanted terms.

In this work, we introduce a general and systematic method to evaluate these terms from the equations of motion describing the physical system at hand (Laser diode rate equations). This case is chosen to highlight the case when the relationship between input and output is non-trivial. This paper is organised as follows. In section 2 we outline the general procedure for expressing the output from the input and define the various distortion terms. In section 3 we apply the procedure in detail to the Single Mode Laser diode order by order up to third and conclude in section 4. The Appendix contains the full expressions of all terms derived in section 3.

II. GENERAL PROCEDURE

The Single Mode (SM) laser diode rate equations are written as:

$$\frac{dN}{dt} = \frac{I}{qV} - \frac{N}{\tau_n} - g(N - N_t)(1 - \epsilon S)S \quad (1)$$

$$\frac{dS}{dt} = \beta \frac{N_t}{\tau_n} - \frac{S}{\tau_{ph}} + g(N - N_t)(1 - \epsilon S)S \quad (2)$$

N represents the electron density (N_t at transparency) and S the photon density. τ_n is the electron spontaneous lifetime and τ_{ph} is the photon lifetime. β is the fraction of spontaneous emission coupled into the lasing mode, ϵ the optical confinement factor, g is the differential gain and ϵ is the gain compression parameter. q is the electron charge, V the volume of the active region and I is the injection current.

a- Step 1 [Elimination of $(N - N_t)$]:

The input is the injection current I and the output is the light intensity represented by S . Since other variable such as N , the electron density, intervene in the SM equations we proceed to the elimination of that variable. When several modes are present (or other intermediate variables) we proceed in an analogous manner by successive elimination until we have a single equation relating input to output.

From (2) we extract the value of $(N - N_t)$ as:

$$(N - N_t) = \left[\frac{\frac{dS}{dt} - \frac{\Gamma\beta N_t}{\tau_n} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S + \frac{\Gamma\beta}{\tau_n}} \right] \quad (3)$$

This can be used to find dN/dt and eliminate completely N from the coupled equations (1) and (2). The result is:

$$\frac{d}{dt} \left[\frac{\frac{dS}{dt} - \frac{\Gamma\beta N_t}{\tau_n} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S + \frac{\Gamma\beta}{\tau_n}} \right] = \frac{I}{qV} - \frac{N_t}{\tau_n} - [g(1 - \epsilon S)S + \frac{1}{\tau_n}] \left[\frac{\frac{dS}{dt} - \frac{\Gamma\beta N_t}{\tau_n} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S + \frac{\Gamma\beta}{\tau_n}} \right] \quad (4)$$

b- Step 2 [Small signal expansion about a static operating point]:

The useful SM Laser diode regime with an injection current modulating the output light intensity. In the small dynamic signal case, $I = I_0 + i$ and $S = S_0 + s$, where I_0 and S_0 are

the static injection current and photon density respectively, whereas, $i = i(t)$ and $s = s(t)$, are the signals of interest. We expand (4) [to n-th order]:

$$i = \{A_1s + B_1s' + C_1s''\} + \{A_2s^2 + B_2ss' + C_2ss'' + D_2[s']^2\} \dots + \{A_ns^n + B_ns^{n-1}s' + C_ns^{n-1}s'' + D_ns^{n-2}[s']^2\} \quad (5)$$

The expansion has this form because the first time derivative is applied to a fraction whose expansion contains n-order terms of the form $s^{n-1}s'$ and s^n . After the derivation we obtain n-order terms of the form $s^{n-1}s''$, $s^{n-2}[s']^2$ and $s^{n-1}s'$. s' and s'' are first and second time derivatives of s . In addition we still have n-order terms of the generic s^n and $s's^{n-1}$ forms coming from the expansion of the right hand side of (4). Hence the general n-order term is $A_ns^n + B_ns^{n-1}s' + C_ns^{n-1}s'' + D_ns^{n-2}[s']^2$ where all the parameters A_n, B_n, C_n, D_n depend on the laser parameters. The appendix contains explicit expressions for some of these coefficients. The absence of a constant term means that no dynamic response i exists when there is no dynamic light excitation [$s=0$].

Incidentally, we have used for simplicity of illustration the SM standard equations that contain two coupled population N (electron) and S (photon) equations in the form of two first-order ordinary differential equations. The method is in fact valid for any number of population equations as long as the elimination procedure of all intermediate variables (or populations) is possible, leaving us with a single equation relating i to s (Equation (5)) or input to output.

The harmonic distortions and intermodulation distortions are calculated from the generalized transfer functions denoted as $H_n(\omega_1, \omega_2, \omega_3 \dots)$. They are obtained from the Fourier transform of the Volterra impulse response h_n in the following way:

$$H_n(j\omega_1, j\omega_2, j\omega_3 \dots) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t_1, t_2, t_3, \dots) e^{-j(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3 \dots)} dt_1 dt_2 dt_3 \dots \quad (6)$$

For instance, the n-th order distortion is given by:

$$M_n(\pm j\omega_1, \pm j\omega_2, \pm j\omega_3 \dots) = 20 \log_{10} \left\{ \frac{|H_n(\pm j\omega_1, \pm j\omega_2, \pm j\omega_3 \dots)|}{2^{n-1} \prod_{m=1}^n |H_1(\pm j\omega_m)|} \right\} \quad (7)$$

c- Step 3 [Method of Harmonic Input allowing the calculation of the Volterra transfer functions H_n]:

The Harmonic Input Method (HIM) allows us to find directly all the H_n 's in the following way:

1. Express i as the sum : $\exp(j\omega_1 t) + \exp(j\omega_2 t) + \exp(j\omega_3 t) \dots$
2. Express s as the sum:

$$s = \sum_{k,l,m \dots = 0}^{\infty} G_{klm \dots} \exp[j(k\omega_1 + l\omega_2 + m\omega_3 \dots)t] \quad (8)$$

The different H_n 's are found by direct identification of the $G_{klm \dots}$ coefficients. For instance, we have:

$$G_{000} = 0, G_{100} = H_1(j\omega_1), G_{110} = H_2(j\omega_1, j\omega_2), \dots \quad (9)$$

III. ORDER BY ORDER DISTORTIONS

Starting from the two laser rate equations, Darcie et al. [6] derived formulae pertaining to the second order, third order and intermodulation distortions for a channel excited by a superposition of two signals with angular frequencies ω_1 and ω_2 . The second and third order distortions are calculated respectively at $2\omega_1$ and $3\omega_1$ whereas the intermodulation distortions are evaluated at $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$. From our formula (7), we can write these distortions as [9, 10]:

a- Second order:

$$M_2(j\omega_1, j\omega_1) = 20 \log_{10} \left\{ \frac{|H_2(j\omega_1, j\omega_1)|}{2|H_1(j\omega_1)H_1(j\omega_1)|} \right\} \quad (10)$$

b- Third order:

$$M_3(j\omega_1, j\omega_1, j\omega_1) = 20 \log_{10} \left\{ \frac{|H_3(j\omega_1, j\omega_1, j\omega_1)|}{4|H_1(j\omega_1)H_1(j\omega_1)H_1(j\omega_1)|} \right\} \quad (11)$$

c- Intermodulation:

$$M_3(j\omega_1, j\omega_1, -j\omega_2) = 20 \log_{10} \left\{ \frac{|H_3(j\omega_1, j\omega_1, -j\omega_2)|}{4|H_1(j\omega_1)H_1(j\omega_1)H_1(-j\omega_2)|} \right\}, \quad (12)$$

$$M_3(j\omega_2, j\omega_2, -j\omega_1) = 20 \log_{10} \left\{ \frac{|H_3(j\omega_2, j\omega_2, -j\omega_1)|}{4|H_1(j\omega_2)H_1(j\omega_2)H_1(-j\omega_1)|} \right\} \quad (13)$$

The HIM allows us to calculate the values of the various H_n 's for $n=1, 2$ and 3 . As an illustration of the procedure, we calculate H_1 and H_2 after performing steps 1 and 2 and having obtained the expansion (5) to the specified order [9, 10]. First, we calculate H_1 after simply using $i = \exp(j\omega t)$ in (5) and:

$$s = \sum_{k=0}^{\infty} G_k \exp[j(k\omega)t] \quad (14)$$

The term G_1 [with the identification of the terms multiplying $\exp(j\omega t)$] obeys the relation:

$$1 = [A_1 + j\omega B_1 - \omega^2 C_1] G_1 \quad (15)$$

where A_1, B_1 and C_1 depend on the laser parameters. Using the HIM [see (9)] we can write:

$$H_1(j\omega) = G_1 = 1/[A_1 + j\omega B_1 - \omega^2 C_1] \quad (16)$$

The modulus of $H_1(j\omega)$ equal to $1/\sqrt{(A_1 - \omega^2 C_1)^2 + (\omega B_1)^2}$ is what Darcie et al. [6] call the small-signal frequency response $R(\omega)$. In order to calculate second order terms, we use $i(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t)$ in equation (5) [truncated to second order] along with:

$$s = \sum_{k,l=0}^{\infty} G_{kl} \exp[j(k\omega_1 + l\omega_2)t] \quad (17)$$

As stated in (9) the various G_{kl} are obtained from the following [9, 10]: $G_{00} = 0$., $G_{10} = H_1(j\omega_1)$, $G_{01} = H_1(j\omega_2)$ and $G_{11} = H_2(j\omega_1, j\omega_2)$. Identification of the terms multiplying

$\exp(j[\omega_1 + \omega_2]t)$ yields:

$$0 = [A_1 + j(\omega_1 + \omega_2)B_1 - (\omega_1 + \omega_2)^2 C_1]G_{11} + [A_2 G_{01} G_{10} + A_2 G_{10} G_{01} + j\omega_1 B_2 G_{01} G_{10} \\ + j\omega_2 B_2 G_{10} G_{01} - \omega_1^2 C_2 G_{01} G_{10} - \omega_2^2 C_2 G_{10} G_{01} - \omega_1 \omega_2 D_2 G_{01} G_{10} - \omega_1 \omega_2 D_2 G_{10} G_{01}] \quad (18)$$

With the use of (16) this can be written as:

$$H_2(j\omega_1, j\omega_2) = G_{11} = \\ -[2A_2 + j(\omega_1 + \omega_2)B_2 - (\omega_1^2 + \omega_2^2)C_2 - 2\omega_1 \omega_2 D_2]H_1(j\omega_1)H_1(j\omega_2)H_1(j[\omega_1 + \omega_2]) \quad (19)$$

The second order distortions are obtained from (10) and (20) once the expressions of A_2, B_2, C_2, D_2 are found from the direct expansion of (4). The Taylor expansion to the third order is made in the Appendix allowing calculations of third order and intermodulation distortions. Also, the values of the various coefficients are given as functions of the laser parameters.

IV. CONCLUSION

In this work, we outline a general and systematic method to evaluate Harmonic and Intermodulation distortions that occur when a signal at a single frequency or a superposition of several signals with different frequencies propagate in a non-linear physical system. The method is generalisable to any order and pertains to cases where a non-trivial relationship exists between input and output exists.

In the case of a SM Laser diode, we have a system of two coupled differential population equations describing the system, nevertheless a general procedure has been described in order to relate the input injection current to the output light intensity. The results obtained in this paper agree with those published in the literature and the methodology highlighted provides a self-contained framework to evaluate distortions in a systematic way.

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Appendix

Procedure:

In order to calculate the various A_n, B_n, C_n and D_n terms (for $n=1, 2, 3$) a Taylor expansion about $S = S_0$ and $s' = 0$ is to be made to third order of equation (4). The following term should be expanded before taking its time derivative:

$$\left[\frac{\frac{dS}{dt} - \frac{\Gamma\beta N_t}{\tau_n} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S + \frac{\Gamma\beta}{\tau_n}} \right] \quad (20)$$

We can simplify the expansion considerably using the following orders of magnitude valid in most single-mode laser diodes: $\beta \cong 0 (\sim 10^{-3} \text{ to } 10^{-4})$, $\tau_n = \infty$, $1 - \epsilon S_0$ is practically 1 and $gS_0 \ll 1/\tau_{ph}$. The term becomes after these considerations:

$$\left[\frac{\frac{dS}{dt} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S} \right] \quad (21)$$

The next term to be expanded to third order is:

$$\left[g(1 - \epsilon S)S + \frac{1}{\tau_n} \right] \left[\frac{\frac{dS}{dt} - \frac{\Gamma\beta N_t}{\tau_n} + \frac{S}{\tau_{ph}}}{g(1 - \epsilon S)S + \frac{\Gamma\beta}{\tau_n}} \right] \quad (22)$$

When we account for the aforementioned orders of magnitude it becomes:

$$\left[\frac{\frac{dS}{dt} + \frac{S}{\tau_{ph}}}{,} \right] \quad (23)$$

The third order expansion of (21) after taking the time derivative gives:

$$\begin{aligned} & [\epsilon/(\tau_{ph}, g)]s' + [1/(\epsilon, gS_0)]s'' + [\epsilon/(\epsilon, gS_0) - 1/(\epsilon, gS_0^2)]([s']^2 + ss'') \\ & + [\epsilon^2/(\epsilon, gS_0) - (3\epsilon S_0 - 2\epsilon^2 S_0^2 - 1)/(\epsilon, gS_0^3)](s^2 s'' + 2s[s']^2) \end{aligned} \quad (24)$$

Adding the contributions from (23) $[s'/, + s/(\tau_{ph},)]$ and term by term identification gives for the various orders:

First order: $A_1 s + B_1 s' + C_1 s''$ where:

$$\begin{aligned} A_1 &= 1/(\tau_{ph},) \\ B_1 &= \epsilon/(\tau_{ph}, g) + 1/, \\ C_1 &= 1/(\epsilon, gS_0) \end{aligned} \quad (25)$$

Second order: $A_2 s^2 + B_2 s s' + C_2 s s'' + D_2 [s']^2$ with:

$$A_2 = 0$$

$$\begin{aligned}
B_2 &= 0 \\
C_2 &= [\epsilon/(\epsilon, gS_0) - 1/(\epsilon, gS_0^2)] \\
D_2 &= C_2
\end{aligned} \tag{26}$$

Third order: $A_3s^3 + B_3s^2s' + C_3s^2s'' + D_3s[s']^2$ with:

$$\begin{aligned}
A_3 &= 0 \\
B_3 &= 0 \\
C_3 &= [\epsilon^2/(\epsilon, gS_0) - (3\epsilon S_0 - 2\epsilon^2 S_0^2 - 1)/(\epsilon, gS_0^3)] \\
D_3 &= 2C_3
\end{aligned} \tag{27}$$

The frequency response:

$$|H_1(j\omega)| = 1 / \left[A_1 \sqrt{(1 - \omega^2 C_1/A_1)^2 + (\omega B_1/A_1)^2} \right] \tag{28}$$

and the second harmonic distortion given by (10) and transformed with the help of (19):

$$M_2(j\omega_1, j\omega_1) = 20 \log_{10} \{ 2\omega_1^2 |C_2 H_1(2j\omega_1)| \} \tag{29}$$

agree with the results of Darcie et al [6].

In order to calculate third order and intermodulation effects, we use eq. (5) truncated to third order and the values of the A_n, B_n, C_n coefficients ($n=1, 2, 3$):

$$i = [A_1s + B_1s' + C_1s''] + C_2[ss'' + [s']^2] + C_3[s^2s'' + 2s[s']^2] \tag{30}$$

Apply the HIM to the above equation with: $i = \exp(j\omega_1 t) + \exp(j\omega_2 t) + \exp(j\omega_3 t)$ and:

$$s = \sum_{k,l,m=0}^{\infty} G_{klm} \exp[j(k\omega_1 + l\omega_2 + m\omega_3)t] \tag{31}$$

Proceeding as before (eq.18) we calculate the G_{klm} coefficients as well as the various Volterra transfer functions:

$$\begin{aligned}
G_{000} &= 0, G_{100} = H_1(j\omega_1), G_{010} = H_1(j\omega_2), G_{001} = H_1(j\omega_3), \\
G_{110} &= H_2(j\omega_1, j\omega_2), G_{101} = H_2(j\omega_1, j\omega_3) \\
G_{011} &= H_2(j\omega_2, j\omega_3) \quad \text{and finally } G_{111} = H_3(j\omega_1, j\omega_2, j\omega_3)
\end{aligned} \tag{32}$$

from eq.(30).

Collecting the terms multiplying $\exp[j(\omega_1 + \omega_2 + \omega_3)t]$ gives:

$$\begin{aligned}
0 = & [A_1 + j(\omega_1 + \omega_2 + \omega_3)B_1 - (\omega_1 + \omega_2 + \omega_3)^2 C_1]G_{111} \\
& - 2C_2(\omega_1 + \omega_2 + \omega_3)^2 [2G_{110}G_{001} + 2G_{101}G_{010} + 2G_{100}G_{011}] \\
& - 3C_3(\omega_1 + \omega_2 + \omega_3)^2 [6G_{100}G_{010}G_{001}]
\end{aligned} \tag{33}$$

Using the above relations between the G_{klm} and the H_n 's and (16) we get:

$$\begin{aligned}
H_3(j\omega_1, j\omega_2, j\omega_3) = & (\omega_1 + \omega_2 + \omega_3)^2 H_1(j[\omega_1 + \omega_2 + \omega_3]) [4C_2 \{ H_2(j\omega_1, j\omega_2) H_1(j\omega_3) \\
& + H_2(j\omega_1, j\omega_3) H_1(j\omega_2) + H_2(j\omega_2, j\omega_3) H_1(j\omega_1) \} \\
& + 18C_3 H_1(j\omega_1) H_1(j\omega_2) H_1(j\omega_3)]
\end{aligned} \tag{34}$$

Using expressions (16) for $H_1(j\omega)$ and (19) for $H_2(j\omega_1, j\omega_2)$ along with eq.(11), (12) and (13) the third order and intermodulation distortions can be evaluated and they agree again with Darcie et al. [6] results.